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The scattering of a plane flexural wave by a sector of a thin elastic plate with a supported edge☆

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Abstract

A solution is constructed of the problem of the diffraction of a plane flexural wave at the vertex of a thin elastic plate cut out in the form of a sector, on the edge of which the condition of a supported edge (hinged support) is specified.

We consider the diffraction of a harmonically time-dependent (this dependence is not indicated below) plane flexural wave

$$u_{inc} = \exp[-ik\cos(\varphi - \varphi_0)] \tag{1}$$

at the vertex of a thin elastic plate S cut out in the form of a sector $0 < \varphi < \Phi$.

The vertical displacement $u = u(r, \varphi)$ of the plate satisfies the equation

$$\Delta^2 u - k^4 u = 0 \tag{2}$$

The wave number k is related to the frequency of vibrations by a well-known equation (Ref. 1, formula (25.8)].

At the plate boundary ∂S, i.e. when $\varphi = 0$, $\varphi = \Phi$, the following boundary condition of a hinged support is specified (see Ref. 1, formula (12.11)

$$u = 0, \quad \partial^2 u/\partial\varphi^2 = 0 \quad \text{when} \quad \varphi = 0, \quad \varphi = \Phi \tag{3}$$

At the vertex of the sector, the Meixner condition

$$u(x, y) = Ax + By + O(r^{1+\alpha}) \quad \text{when} \quad r \to 0 \tag{4}$$

must be satisfied, where $\alpha > 0$, and here this estimate can be differentiated.

As will be shown in the Appendix, for Eq. (2) and boundary conditions (3) this condition is equivalent to the generally accepted requirement of the boundedness in the vicinity of the vertex of the potential energy W of the field of vertical displacements $u(r, \varphi)$, i.e. convergence at the vertex of the sector of the integral

$$W = \iint E(r, \varphi)\, r\, dr\, d\varphi \tag{5}$$

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where $E(r, \varphi)$ is the average value, with respect to the period of vibrations, of the potential energy density. Below, in proving the theorem of uniqueness and constructing the solution, we will use condition (4).

At infinity, we will require the condition of limiting absorption to be satisfied, that is, we will represent the vertical displacement in the form

$$u = u_{GO} + u_{\text{dif}}$$

where u_{GO} is the sum of the incident wave and all its reflections from the sides of the plate (including multiple reflections when $\Phi < \pi$). The function u_{dif} is the difference between the exact solution u and its geometrical-optical approximation u_{GO}. It is required that, when $0 < \text{Im } k = \delta$, where $\delta \ll 1$ (which corresponds to small absorption of vibrations of the plate), the wave u_{dif} is damped exponentially as $r \to \infty$:

$$u_{\text{dif}} = O(\exp(-\gamma r \text{ Im}k))$$

for some $\gamma > 0$.

Thus, the problem is to determine the field u satisfying the condition of regularity (4) at the vertex, the condition of limiting absorption at infinity and boundary conditions (3).

We will prove the uniqueness of the solution of this problem. We note, first of all, that it follows from condition (3) that $\Delta u = 0$ on ∂S. Suppose, now, that u is the difference of the two solutions of Eq. (1) that satisfy conditions (3), Meixner's condition at the vertex and the condition of limiting absorption at infinity, and $\bar{u}$ is its complex-conjugate function. Then

$$\int_{\varepsilon}^{\infty}\int_{0}^{\Phi} \Delta u \Delta \bar{u} r dr d\varphi = \int_{L(\varepsilon)} \left[\frac{\partial \Delta u}{\partial r}\bar{u} - \frac{\partial \bar{u}}{\partial r}\Delta u\right] r d\varphi + k^4 \int_{\varepsilon}^{\infty}\int_{0}^{\Phi} u \bar{u} r dr d\varphi$$

where $L(\varepsilon)$ is the arc $r = \varepsilon$, $0 < \phi < \Phi$. The integral over the faces of the plate is equal to zero by virtue of boundary conditions (3); the integral over $L(\varepsilon)$ is of the order of ε^{α}. Taking the limit as $\varepsilon \to 0$, we obtain the equation

$$\int_S |\Delta u|^2 r dr d\varphi = k^4 \int_S |u|^2 r dr d\varphi$$

the left-hand side of which is positive, while the right-hand side has a non-zero imaginary part. From this it follows that $u = 0$.

We will construct the solution of the problem. We will denote by $u_0(kr, \varphi)$ the solution of the scalar problem of diffraction by a wedge of the incident wave (1.1), i.e. the solution of Helmholtz equation $\Delta u_0 + k^2 u_0 = 0$ in the angular region $r > 0$, $0 < \varphi < \Phi$, which, when $\varphi = 0$ and $\varphi = \Phi$, satisfies Dirichlet's boundary condition $u|_{\partial S} = 0$, the condition of boundedness at the vertex $u = O(r^p)$ when $p = 0$ and $r \to 0$, and the condition of limiting absorption for the field $u_{\text{dif}} = u_0 - u_{GO}$. The function $u_0(kr, \varphi)$ is described by the well-known Sommerfeld integral[2]

$$\begin{aligned} u_0(kr, \varphi) &= \frac{1}{2\pi i}\int_{\gamma} \exp(-ikr\cos\alpha) H(\alpha + \varphi, \varphi_0, \Phi) d\alpha \\ H(\varphi, \varphi_0, \Phi) &= \frac{\pi}{2\Phi}\left[\text{ctg}\frac{\pi(\varphi - \varphi_0)}{2\Phi} - \text{ctg}\frac{\pi(\varphi + \varphi_0)}{2\Phi}\right] \end{aligned} \tag{6}$$

The contour γ consists of two loops, γ_1 and γ_2, and is shown in Fig. 1.

Since the quantity u_0 vanishes on S and satisfies the Helmholtz equation, it satisfies the boundary conditions (3). However, when $\Phi > \pi$, it does not satisfy condition (4) of regularity at the vertex.

In fact, u_0 can be expanded in a converging series (see, for example, Ref. 3)

$$u_0 = \frac{4\pi}{\Phi}\sum_{m=1}^{\infty} \exp\left(-\frac{im\pi^2}{\Phi}\right) \sin\frac{m\pi\varphi}{\Phi} \sin\frac{m\pi\varphi_0}{\Phi} J_{m\pi/\Phi}(kr) \tag{7}$$

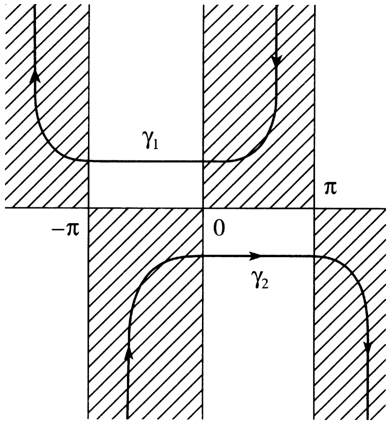

Fig. 1.

We will introduce the notation

$$\chi(\varphi, \varphi_0, \Phi) = \exp\left(-i\frac{\pi^2}{\Phi}\right)(kr)^{2\pi/\Phi}\sin\frac{\pi\varphi}{\Phi}\sin\frac{\pi\varphi_0}{\Phi}$$

From relation (7) it follows that

$$u_0 = \frac{4\pi}{\Phi}(\chi(\varphi, \varphi_0, \Phi) + \chi(\varphi, \varphi_0, 2\Phi)) + O((kr)^{\pi/\Phi+2}) + O((kr)^{3\pi/\Phi}),$$

It can be seen that, when $\Phi > \pi$, Meixner's condition (4) is not satisfied. In order to satisfy this condition, we will put

$$u = u_0 + (AH^{(1)}_{\pi/\Phi}(kr) + BH^{(1)}_{\pi/\Phi}(ikr))\sin\frac{\pi\varphi}{\Phi}\sin\frac{\pi\varphi_0}{\Phi} \tag{8}$$

where $H^{(1)}_\nu(\xi)$ is the Hankel function of the first kind. It is obvious that this expression satisfies Eq. (2), boundary conditions (3) and the condition of limiting absorption at infinity. We will select constants A and B in such a way that, in expansion (8) in powers of kr, there is no term with the factor $(kr)^{\pi/\Phi}$. As a result we obtain

$$A = -B\exp\left(-i\frac{\pi^2}{2\Phi}\right), \quad B = \frac{\pi 2^{\pi/\Phi+2}\Gamma(\pi/\Phi+1)}{\Phi[1 - i\,\mathrm{tg}(\pi^2/(2\Phi))]} \tag{9}$$

Then

$$u = \frac{4\pi}{\Phi}\exp\left(-i\frac{\pi^2}{\Phi}\right)(kr)^{2\pi/\Phi}\sin\frac{2\pi\varphi}{\Phi}\sin\frac{2\pi\varphi_0}{\Phi} + O((kr)^{2-\pi/\Phi})$$

From this it follows that the function u satisfies the condition of regularity at the vertex when $\Phi < 2\pi$. When $\Phi = 2\pi$ we have

$$u = -2iky\sin\varphi_0 + O(kr)^{3/2} \tag{10}$$

It is obvious that function (10) also satisfies this condition. Thus, the solution of the problem with $\Phi < \pi$ is identical with the solution (6) of the scalar problem of diffraction by a wedge, and when $\Phi > \pi$ it has the form (8), (9).

Appendix. The average value of the potential energy density over a period (see Ref. 1, formula (11.6)) can be written, apart from a constant coefficient, in the form

$$E(u) = (1-\sigma)(|u_{xx}|^2 + 2|u_{xy}|^2 + |u_{yy}|^2) + \sigma|\Delta u|^2 \tag{11}$$

where σ is Poisson's ratio.

It is obvious that, if Meixner's condition (4) is satisfied, then integral (5) converges in the vicinity of the vertex. We will show the validity of the reverse assertion: from the boundedness of the potential energy in the vicinity of the vertex it follows that of condition (4) is satisfied.

According to general results,[4,5] the function u can be expanded in the vicinity of the vertex in an asymptotic series allowing of differentiation

$$u(r,\varphi) \approx \sum_{m,n=0}^{\infty} r^{q_m+2n} \sum_{l=0}^{N_m-1} f_{m,n,l}(\varphi)(\ln r)^l \quad \text{when} \quad r \to 0 \tag{12}$$

where $q_m < q_{m+1}$, and, as a consequence of condition (3),

$$f_{m,n,l}(\varphi) = \frac{\partial^2 f_{m,n,l}(\varphi)}{\partial \varphi^2} = 0 \quad \text{when} \quad \varphi = 0, \quad \varphi = \Phi$$

The main term E_0 of the asymptotic expansion of the energy density E as $r \to 0$ is defined by the main term $u_0 = r^{q_0}(\ln r)^{N_0-1} f_{0,0,(N_0-1)}\varphi$ of expansion (12)

$$E_0 = (\ln r)^{N_0-1} E(r^{q_0} f_{0,0,(N_0-1)}(\varphi)) \tag{13}$$

When $r \to 0$, the second derivatives of the function u_0 behave as $r^{q_0-2}(\ln r)^{N_0-1}$. Therefore

$$E_0 = r^{2q_0-4}(\ln r)^{2(N_0-1)} A(f_{0,0,(N_0-1)}(\varphi))$$

where $A(f(\varphi))$ is a non-negative definite quadratic form of the real and imaginary parts of the function $f(\varphi)$ and its first and second derivatives, which vanish only when $q_0 = 1$ and

$$f(\varphi) = f_{0,0,(N_0-1)}(\varphi) = P\sin\varphi + Q\cos\varphi$$

However, in this case, it follows from boundary condition (13) that $P = Q = 0$. In all remaining cases

$$A(f_{0,0,(N_0-1)}(\varphi)) > 0$$

and for the integral (5) to converge as $r \to 0$ it is necessary that $q_0 > 1$, i.e. it is necessary for Meixner's condition (4) to be satisfied.

Similarly, it can be proved that Meixner's condition is equivalent to the requirement that the energy flux through an arc of radius r_0 must tend to zero as $r_0 \to 0$.

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